Algorithm for multiplying Schubert classes

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Abstract

Based on the multiplicative rule of Schubert classes obtained in [Du_3], we present an algorithm computing the product of two arbitrary Schubert classes in a flag variety G/H, where G is a compact connected Lie group and $H \subset G$ is the centralizer of a one-parameter subgroup in G.

Since all Schubert classes on G/H constitute an basis for the integral cohomology $H^*(G/H)$, the algorithm gives a method to compute the cohomology ring $H^*(G/H)$ independent of the classical spectral sequence method due to Leray [L_1,L_2] and Borel [Bo_1, Bo_2].

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1 Introduction

This paper presents an algorithm computing the integral cohomology ring of a flag manifold G/H, where G is a compact connected Lie group and $H \subset G$ is the centralizer of a one-parameter subgroup.

The determination of the integral cohomology of a topological space is a classical problem in algebraic topology. However, since a flag manifold G/H is canonically an algebraic variety whose Chow ring is isomorphic to the integral cohomology $H^*(G/H)$, a complete description for the ring $H^*(G/H)$ is also of fundamental importance to the algebraic intersection theory of G/H ([K,S_2]).

In general, an entire account for the integral cohomology $H^*(X)$ of a space X leads to two inquiries.

Problem A. Specify an additive basis for the graded abelian group $H^*(X)$ that encodes the geometric formation of X (e.g. a cell decomposition of X).

Problem B. Determine the table of multiplications between these base elements.

It is plausible that if X is a flag manifold G/H, a uniform solution to Problem A is afforded by the *Basis Theorem* from the *Schubert enumerative calculus* [S_2]. This was originated by Ehresmann for the Grassmannians $G_{n,k}$ of k-dimensional subspaces in $\mathbb{C}^n$ in 1934 [E], extended to the case where G is a matrix group by Bruhat in 1954, and completed for all compact connected Lie groups by Chevalley in 1958 [Ch_2]. We briefly recall the result.

Let W and W' be the Weyl groups of G and H respectively. The set W/W' of left cosets of W' in W can be identified with the subset of W:

$$\overline{W} = \{w \in W \mid l(w_1) \geq l(w) \text{ for all } w_1 \in wW'\},$$

where $l : W \to \mathbb{Z}$ is the length function relative to a fixed maximal torus T in G [BGG, 5.1. Proposition]. The key fact is that the space G/H admits a canonical decomposition into cells indexed by elements of $\overline{W}$

$$G/H = \underset{w\in\overline{W}}{\cup} X_w, \quad \dim X_w = 2l(w), \tag{1.1}$$

with each cell X_w the closure of an algebraic affine space, known as a *Schubert variety* in G/H [Ch_2, BGG].

Since only even dimensional cells are involved in the decomposition (1.1), the set of fundamental classes $[X_w] \in H_{2l(w)}(G/H)$, $w \in \overline{W}$, forms an additive basis of $H_*(G/H)$. The cocycle class $P_w \in H^{2l(w)}(G/H)$, $w \in \overline{W}$, defined by the Kronecker pairing as $\langle P_w, [X_u]\rangle = \delta_{w,u}$, $w, u \in \overline{W}$, is called the *Schubert class corresponding to* w. The solution to Problem A can be stated in (cf. [BGG])

Basis Theorem. *The set of Schubert classes $\{P_w \mid w \in \overline{W}\}$ constitutes an additive basis for the ring $H^*(G/H)$.*

One of the direct consequences of the basis Theorem is that the product of two arbitrary Schubert classes can be expressed in terms of Schubert classes. Precisely, given $u, v \in \overline{W}$, one has the expression

$$P_u \cdot P_v = \sum_{l(w)=l(u)+l(v), w\in\overline{W}} a_{u,v}^w P_w,\ a_{u,v}^w \in \mathbb{Z}$$

in $H^*(G/H)$. Thus, in the case of $X = G/H$, Problem B has a concrete form.

Problem B'. *Determine the structure constants $a_{u,v}^w$ of the ring $H^*(G/H)$ for $w, u, v \in \overline{W}$ with $l(w) = l(u) + l(v)$.*

Originated in the pioneer works of Schubert on enumerative geometry from 1874 and spurred by Hilbert's fifteenth problem, the study of Problem B' has a long and outstanding history even for the very special case $G = U(n)$ and $H = U(k) \times U(n-k)$, where $U(n)$ is the unitary group of rank n (cf. [K]). The corresponding flag manifold G/H is *the Grassmannian* $G_{n,k}$ of k-planes through the origin in $\mathbb{C}^n$, and the solution to Problem B' is given by the classical *Pieri formula*[1] and the *Littlewood-Richardson rule*[2]. We refer to the articles [KL] by Kleiman-Laksov and [St] by Stanley for full expositions of these results respectively from geometric approach and from combinatorial view-point.

During the past half century, many achievements have been made in extending the knowledge on the $a_{u,v}^w$ from the $G_{n,k}$ to flag manifolds of other types. See [Ch$_1$], [Mo], [BGG], [D], [LS$_2$], [HB], [KK], [Wi], [BS$_1$-BS$_3$], [S$_2$], [PR$_1$- PR$_3$], [Bi].

Early in 1953, Borel introduced a method to compute the cohomology algebra $H^*(G/H; \mathbb{R})$ (with real coefficients) using spectral sequence technique [Bo$_1$, Bo$_2$, B, TW, W]. In the results so obtained the algebra $H^*(G/H; \mathbb{R})$ was characterized algebraically in terms of generators-relations, in which the basis theorem that implies the geometric structure of the space G/H was absent[3]. In recent years, in order to recover from Borel's description of the algebra $H^*(G/H; \mathbb{R})$ the polynomial representatives of Schubert classes so that explicit computation for the $a_{u,v}^w$ is possible, various theories of *Schubert polynomials* were developed for the cases where G is a matrix group and $H \subset G$ is a maximal torus (cf. [S$_2$], [LS$_1$], [Be], [BH], [BJS], [FK], [FS], [Fu], [LPR], [Ma]).

Combining the ideas of the Bott-Samelson resolutions of Schubert varieties [BS, Han] and the enumerative formula on a twisted product of 2 spheres developed in [Du$_2$], the first author obtained in [Du$_3$] a formula expressing the structure constant $a_{u,v}^w$ in terms of Cartan numbers of G. It was also announced in [Du$_3$]

[1]In order to find a formula for the degrees of Schubert varieties on the Grassmanian, Schubert himself developed a special case of the Pieri formula [K].

[2]Classically, the Littlewood-Richardson rule describes the multiplicative rule of Schur symmetric functions. It was first stated by Littlewood and Richardson in 1934 [LR] and completly proofs appeared in the 70's (see "Note and references" in [M, p.148]). Lesieur noticed in 1947 [L] that the multiplicative rule of Schubert classes in the Grassmanian formly coincides with that of Schur functions. That is, the Littlewood-Richardson rule can also be considered as the rule for multiplying Schubert classes in the Grassmanians.

[3]In the intersection theory, the basis Theorem is important for it guarantees that the rational equivalence class of a subvariety in G/H can be expressed in term of the base elements and therefore, the intersection multiplicities of arbitrary subvarities in G/H can be computed in terms of the $a_{u,v}^w$.

that, based on the formula, a program to compute the $a^w_{u,v}$ can be compiled. This paper is devoted to explain the algorithm in details.

Consequently, the algorithm gives a method to compute the integral cohomology ring $H^*(G/H)$ independent of the classical spectral sequence method due to Leray [L_1,L_2] and Borel [Bo_1, Bo_2]. It has also served the purpose to indicate our further programs computing Steenrod operations on G/H [DZ], and multiplying Demazure basis (resp. Grothendieck basis) in the Grothendieck cohomology of G/H [Du_4].

The paper is so arranged. In Section 2 we recall the formula for the $a^w_{u,v}$ from [Du_3]. In Section 3 we resolve Problem B' into two algorithms entitled "*Decompositions*" and "*L-R coefficients*". The functions of the algorithms are implemented respectively in section 4 and 5.

Explicit computation in the cohomology (i.e. the Chow ring) of such classical spaces as flag varieties is not only required by the effective computability of problems from enumerative geometry [K], but also related to many problems from geometry and topology [IM, H, Du_1]. In order to demonstrate that our algorithm is effective, samples of computational results from the program are explained and tabulated in Section 6.

It is worth to mention that there have been excellent codes for multiplying Schubert classes in the Grassmannians $G_{n,k}$ (cf. the programs SYMMETRICA at Bayreuth

<http://www. mathe2.uni-bayeuth.de>;

the programs ACE at Marne La Valle

<http://phalanstere.univ-mlv.fr>,

and the program LITTLWOOD-RICHARDSON CALCULATOR at Aarhus

<http://home.imf.au.dk/abuch>).

Instead of being type-specific, our program applies uniformly to all G/H.

2 The formula

This section recalls the formula for the $a^w_{u,v}$ from [Du_3]. A few preliminary notations will be needed. Throughout this paper G is a compact connected Lie group with a fixed maximal torus T. We set $n = \dim T$.

Equip the Lie algebra $L(G)$ of G with an inner product $(,)$ so that the adjoint representation acts as isometries of $L(G)$. The *Cartan subalgebra* of G is the Euclidean subspace $L(T)$ of $L(G)$ [Hu, p.80].

The restriction of the exponential map $\exp : L(G) \to G$ to $L(T)$ defines a set $D(G)$ of $m = \frac{1}{2}(\dim G - n)$ hyperplanes in $L(T)$, i.e. the set of *singular hyperplanes* through the origin in $L(T)$. These planes divide $L(T)$ into finitely many convex cones, called the *Weyl chambers* of G. The reflections σ of $L(T)$ in the these planes generate *the Weyl group* W of G.

Fix, once and for all, a regular point $\alpha \in L(T) \setminus \bigcup_{L \in D(G)} L$ and let $\Delta = \{\beta_1, \cdots, \beta_n\}$ be the set of simple roots relative to α [Hu, p.47]. For a $1 \leq i \leq n$, write $\sigma_i \in W$ for the reflection of $L(T)$ in the singular plane $L_{\beta_i} \in D(G)$ corresponding to the root β_i. The σ_i are called *simple reflections* [Hu, 42].

Recall that for $1 \leq i, j \leq n$, *the Cartan number* $\beta_i \circ \beta_j =: 2(\beta_i, \beta_j)/(\beta_j, \beta_j)$ is always an integer (only $0, \pm 1, \pm 2, \pm 3$ can occur) [Hu, p.39, p.55].

It is known that the set of simple reflections $\{\sigma_i \mid 1 \leq i \leq n\}$ generates W. That is, any $w \in W$ admits a factorization of the form

$$w = \sigma_{i_1} \circ \cdots \circ \sigma_{i_k}, . \tag{2.1}$$

Definition 1. The *length* $l(w)$ of a $w \in W$ is the least number of factors in all decompositions of w in the form (2.1). The decomposition (2.1) is said *reduced* if $k = l(w)$.

If (2.1) is a reduced decomposition, the $k \times k$ (strictly upper triangular) matrix $A_w = (a_{s,t})$ with

$$a_{s,t} = \left\{ \begin{array}{c} 0 \text{ if } s \geq t; \\ -\beta_{i_t} \circ \beta_{i_s} \text{ if } s < t \end{array} \right.$$

is called *the Cartan matrix of w associated to the decomposition* (2.1).

Let $\mathbb{Z}[x_1, \cdots, x_k] = \oplus_{r \geq 0} \mathbb{Z}[x_1, \cdots, x_k]^{(r)}$ be the ring of integral polynomials in $x_1, \cdots, x_k$, graded by $\mid x_i \mid = 1$.

Definition 2. Given an $k \times k$ strictly upper triangular integer matrix $A = (a_{i,j})$, the *triangular operator* associated to A is the homomorphism $T_A : \mathbb{Z}[x_1, \cdots, x_k]^{(k)} \to \mathbb{Z}$ defined recursively by the following *elimination laws*.

1) if $h \in \mathbb{Z}[x_1, \cdots, x_{k-1}]^{(k)}$, then $T_A(h) = 0$;

2) if $k = 1$ (consequently $A = (0)$), then $T_A(x_1) = 1$;

3) if $h \in \mathbb{Z}[x_1, \cdots, x_{k-1}]^{(k-r)}$ with $r \geq 1$, then

$$T_A(hx_k^r) = T_{A'}(h(a_{1,k}x_1 + \cdots + a_{k-1,k}x_{k-1})^{r-1}),$$

where A' is the ($(k-1)\times(k-1)$ strictly upper triangular) matrix obtained from A by deleting the k^{th} column and the k^{th} row.

By additivity, T_A is defined for every $f \in \mathbb{Z}[x_1,\cdots,x_k]^{(k)}$ using the unique expansion $f = \Sigma h_r x_k^r$ with $h_r \in \mathbb{Z}[x_1,\cdots,x_{k-1}]^{(k-r)}$.

Example. Definition 2 implies an effective algorithm to evaluate T_A.

For $k=2$ and $A_1 = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$, then $T_{A_1} : \mathbb{Z}[x_1,x_2]^{(2)} \to \mathbb{Z}$ is given by

$$\begin{aligned} &T_{A_1}(x_1^2) = 0,\\ &T_{A_1}(x_1x_2) = T_{A_1'}(x_1) = 1 \text{ and}\\ &T_{A_1}(x_2^2) = T_{A_1'}(ax_1) = a. \end{aligned}$$

For $k=3$ and $A_2 = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$, then $A_2' = A_1$ and $T_{A_2} : \mathbb{Z}[x_1,x_2,x_3]^{(3)} \to \mathbb{Z}$ is given by

$$T_{A_2}(x_1^{r_1}x_2^{r_2}x_3^{r_3}) = \begin{cases} 0, \text{ if } r_3 = 0 \text{ and} \\ T_{A_1}(x_1^{r_1}x_2^{r_2}(bx_1+cx_2)^{r_3-1}), \text{ if } r_3 \geq 1, \end{cases}$$

where $r_1+r_2+r_3 = 3$, and where T_{A_1} is calculated in the above.

Assume that $w = \sigma_{i_1}\circ\cdots\circ\sigma_{i_k}$, $1 \leq i_1,\cdots,i_k \leq n$, is a reduced decomposition of $w \in \overline{W}$, and let $A_w = (a_{s,t})_{k\times k}$ be the associated Cartan matrix. For a subset $L = [j_1,\cdots,j_r] \subseteq [1,\cdots,k]$ we put $\mid L \mid = r$ and set

$$\sigma_L = \sigma_{i_{j_1}}\circ\cdots\circ\sigma_{i_{j_r}} \in W; \quad x_L = x_{j_1}\cdots x_{j_r} \in \mathbb{Z}[x_1,\cdots,x_k].$$

The solution to Problem B' is (cf. [Du$_3$])

The formula. *If $u,v \in \overline{W}$ with $l(w) = l(u)+l(v)$, then*

$$a_{u,v}^w = T_{A_w}[(\sum_{|L|=l(u),\ \sigma_L=u} x_L)(\sum_{|K|=l(v),\ \sigma_K=v} x_K)],$$

where $L,K \subseteq [1,\cdots,k]$.

The subsequent sections are devoted to clarify the algorithm implicitly contained in the formula.

3 The structure of the algorithm

Let $L(T)$ be the Cartan subalgebra of G and let $\Delta = \{\beta_1, \cdots, \beta_n\} \subset L(T)$ be the set of simple roots of G relative to the regular point $\alpha \in L(T)$ (cf. Section 2). The *Cartan matrix* of G is the $n \times n$ integral matrix $C = (c_{ij})_{n\times n}$ defined by

$$c_{ij} = 2(\beta_i, \beta_j)/(\beta_j, \beta_j),\ 1 \leq i, j \leq n.$$

It is well known that (cf. [Hu])

Fact 1. *All simply connected compact semi-simple Lie groups are classified by their Cartan matrices.*

For a subset $K = [i_1, \cdots, i_d] \subset [1, \cdots, n]$ let $b \in L(T)\backslash\{0\}$ be a point lying *exactly* in the singular hyperplanes $L_{\beta_{i_1}}, \cdots, L_{\beta_{i_d}}$; namely,

$$b \in \bigcap_{i\in K} L_{\beta_i} \backslash \bigcup_{j\in J} L_{\beta_j}\ (\in L(T)\backslash \bigcup_{j\in J} L_{\beta_j} \text{ if } K = \emptyset) \tag{3.1}$$

where J is the complement of K in $[1, \cdots, n]$. Denote by H_K the centralizer of the 1-parameter subgroup $\{\exp(tb) \mid t \in \mathbb{R}\}$ in G. It can be shown that (cf. [BHi, 13.5-13.6]))

Fact 2. *The isomorphism type of the Lie group H_K depends only on the subset K and not on a specific choice of b in (3.1). Further, every centralizer H of a one-parameter subgroup in G is conjugate in G to one of the subgroups H_K.*

By Fact 2 we may assume that H is of the form H_K for some $K \subset [1, \cdots, n]$. Summarizing Fact 1 and 2 we have

Lemma 1. *A complete set of numerical invariants required to determine a flag manifold G/H consists of*

1) a Cartan matrix $C = (c_{ij})_{n\times n}$ (to determine G);

2) a subset $K = [i_1, \cdots, i_d] \subset [1, \cdots, n]$ (to specify $H \subset G$).

The implementation of our program essentially consists of two algorithms, whose functions may be briefed as follows.

Algorithm A. *Decompositions.*
Input: *A Cartan matrix $C = (c_{ij})_{n\times n}$, and a subset $K \subset [1, \ldots, n]$.*
Output: *The coset $\overline{W}$ being presented by a reduced decomposition for each $w \in \overline{W}$.*

Remark 1. In [Ste, Section 1] Stembridge described an algorithm for the problem of *finding a reduced decomposition for a given* $w \in W$. This requests less than what Algorithm A concerns.

Algorithm B. *L-R coefficients.*
Input: u, v, $w \in \overline{W}$ *with* $l(u) + l(v) = l(w)$.
Output: $a_{u,v}^{w} \in \mathbb{Z}$.

The details of these algorithms will be given respectively in the coming two sections.

It is clear from the above discussion that our algorithms reduce the structure constants $a_{u,v}^{w}$ directly to the Cartan matrix $C = (c_{ij})_{n\times n}$ and the subset $K \subset [1, \dots, n]$: the simplest and minimum set of constants *by which all flag manifolds* G/H *are classified* (cf. Lemma 1). Because of this feature it is functional equally for computations in all G/H.

4 Algorithm A

We show in **4.1** the fashion by which the Weyl groups $W' \subset W$ arise from the Cartan matrix $C = (c_{ij})_{n\times n}$ and the subset $K \subset [1, \cdots, n]$. In **4.2** a numerical representation for W is introduced. Based on the terminologies developed in **4.1** and **4.2**, Algorithm A is given in **4.3**.

4.1. Constructing the Weyl groups $W' \subset W$ from the Cartan matrix. Let Γ be the free $\mathbb{Z}$-module with n generators $\omega_1, \cdots, \omega_n$, and let $Aut(\Gamma)$ be the group of automorphisms of Γ.

Given a Cartan matrix $C = (c_{ij})_{n\times n}$ of a Lie group G with rank n, define n endomorphisms σ_k of Γ (in term of Cartan numbers) by

$$\sigma_k(\omega_i) = \{ \begin{array}{l} \omega_i \text{ if } k \neq i; \\ \omega_i - \Sigma_{1\leq j\leq n} c_{ij}\omega_j \text{ if } k = i \end{array}, \ 1 \leq k \leq n. \tag{4.1}$$

It is straightforward to verify that $\sigma_k^2 = Id$. In particular, $\sigma_k \in Aut(\Gamma)$.

Lemma 2. *The subgroup of* $Aut(\Gamma)$ *generated by* $\sigma_1, \cdots, \sigma_n$ *is isomorphic to* W, *the Weyl group of* G.

For a subset $K \subset [1, \cdots, n]$, *the subgroup* W' *of* W *generated by* $\{\sigma_i \mid i \in K\}$ *is isomorphic the Weyl group of* H_K (cf. Section 3).

Proof (cf. Proof of Theorem 1 in [DZZ]). Let t be the real vector space spanned by $\omega_1, \cdots, \omega_n$; namely, $t = \Gamma \otimes \mathbb{R}$. In term of the Cartan matrix $C = (c_{ij})_{n\times n}$ we introduce in t the vectors $\beta_1, \cdots, \beta_n$ by

$$\beta_i = c_{i1}\omega_1 + \cdots + c_{in}\omega_n,$$

and define an Euclidean metric on t by

$$2(\beta_i, \tfrac{\beta_j}{(\beta_j,\beta_j)}) = c_{ij};\ (\beta_1, \beta_1) = 1. \tag{4.2}$$

Then

(a) t can be identified with the Cartan subalgebra $L(T)$ of G under which the vectors $\beta_1, \cdots, \beta_n$ corresponds to the set Δ of simple roots of G (cf. Section 2);

(b) with respect to the metric (4.2), the induced action of σ_k on $t = L(T)$ is the reflection in the hyperplane L_{β_k} perpendicular to the β_k;

(c) under the identification $t = L(T)$ specified in (a), the basis $\omega_1, \cdots, \omega_n$ of Γ agrees with the set of *the fundamental dominant weights* relative to Δ [Hu, p.67] (Geometrically, positive multiples of $\omega_1, \cdots, \omega_n$ form the edges of the Weyl chamber in $L(T)$ corresponding to Δ).

Lemma 2 follows directly from (b) and (c).□

4.2. A numerical representation of Weyl groups In the theory of Lie algebras the vector $\delta = \omega_1 + \cdots + \omega_n \in \Gamma \subset t = \Gamma \otimes \mathbb{R}$ is well known as a *strongly dominant weight* [Hu, p.30]. For a $w \in W$ consider the expression in Γ

$$w(\delta) = b_1\omega_1 + \cdots + b_n\omega_n,\ b_i \in \mathbb{Z}.$$

Definition 3. The correspondence $b : W \to \mathbb{Z}^n$ by $b(w) = (b_1, \cdots, b_n)$ will be called *the numerical representation of W*.

Lemma 3. *The numerical representation $b : W \to \mathbb{Z}^n$ is faithful and satisfies $b_i \neq 0$ for all $w \in W$ and $1 \leq i \leq n$.*

Proof. By (c) in the proof of Lemma 2, $\delta \in t$ is a regular point in the Weyl chamber determined by Δ. Lemma 3 comes from the geometric fact that the action of the Weyl group W on the orbit of any regular point is simply transitive.□

The formula (4.1), together with additivity of the σ_k, is sufficient to compute the coordinates of $b(w)$ from the Cartan numbers and any decomposition of $w \in W$ into products of the σ_i, as the following algorithm shows.

Algorithm 1. Computing $b(w)$.
Input: A sequence $1 \leq i_1, \cdots, i_m \leq n$.
Output: $b(w)$ for $w = \sigma_{i_1} \circ \cdots \circ \sigma_{i_m}$.
Procedure: Begin with the sum $p_0 = \omega_1 + \cdots + \omega_n$.
Step 1. Substituting in p_0 the term ω_{i_m} by $\omega_{i_m} - \Sigma_{1\leq j\leq n} c_{i_m j}\omega_j$ to get p_1;
Step 2. Substituting in p_1 the term $\omega_{i_{m-1}}$ by $\omega_{i_{m-1}} - \Sigma_{1\leq j\leq n} c_{i_{m-1} j}\omega_j$ to get p_2;
$\vdots$

Step m. Substituting in p_{m-1} the term ω_{i_1} by $\omega_{i_1} - \Sigma_{1\leq j\leq n} c_{i_1 j}\omega_j$ to get p_m;
Step m+1. If $p_m = b_1\omega_1 + \cdots + b_n\omega_n$ then $b(w) = (b_1, \cdots, b_n)$.

We conclude this subsection with two useful properties of the numerical representation of a Weyl group given in Definition 3. Let $l : W \to \mathbb{Z}$ be the length function on W. As in Section 1 we identify $\overline{W}$ with the subset of W

$$\overline{W} = \{w \in W \mid l(w) \leq l(u) \text{ for all } u \in wW'\}.$$

Lemma 4. *Let $w \in W$ be with $b(w) = (b_1, \cdots, b_n)$ and $b(w^{-1}) = (\overline{b}_1, \cdots, \overline{b}_n)$. Then*

(i) $l(\sigma_i w) = l(w) - 1$ if and only if $b_i < 0$;

(ii) $w \in \overline{W}$ if and only if $\overline{b}_i > 0$ for all $i \in K$.

Proof. The metric on $L(T)$ yields the relations

$$(\omega_i, \beta_j/(\beta_j, \beta_j)) = \delta_{ij} \tag{4.3}$$

between the simple roots β_j and the corresponding fundamental dominant weights ω_i [Hu, p.67]. By [BGG, 2.3 Corollary], $l(\sigma_i w) = l(w) - 1$ if and only if $(w(\delta), \beta_i) < 0$. The latter is equivalent to $b_i < 0$ in views of (4.3) and $w(\delta) = b_1\omega_1 + \cdots + b_n\omega_n$. This verifies (i).

Similarly, assertion (ii) follows from the following alternative description for $\overline{W}$ (cf. [BGG, 5.1. Proposition, (iii)])

$$\overline{W} = \{w \in W \mid (w^{-1}(\delta), \beta_i) > 0 \textit{ for all } i \in K\}.\square$$

4.3. Construction of the coset $\overline{W} = W/W'$. Let $l : W \to \mathbb{Z}$ be the length function on W. We put $\overline{W}^k = \{w \in \overline{W} \mid l(w) = k\}$, $k = 0, 1, 2, \cdots$. Then, as is clear, $\overline{W} = \coprod_{k\geq 0} \overline{W}^k$. The problem concerned by Algorithm A may be reduced to

Problem C. *Enumerate elements in $\overline{W}^k$ (i.e. in $\overline{W}$), $k \geq 0$, by their reduced decompositions.*

Before presenting Algorithm A (i.e. the solution to Problem C) we note that

(4.4) If the set $\overline{W}^k$ is given in term of certain reduced decompositions of its elements, then $\overline{W}^k$ becomes an ordered set with the order specified by

$$\sigma_{i_1} \circ \cdots \circ \sigma_{i_k} < \sigma_{j_1} \circ \cdots \circ \sigma_{j_k}$$

if there exists some $s \leq k$ such that $i_t = j_t$ for all $t < s$ but $i_s < j_s$.

(4.5) If X and Y are two ordered sets, then the product $X \times Y$ is furnished with the canonical order as:

"$(x, y) < (x', y')$ if and only if $x < x'$ or $x = x'$ but $y < y'$".

The solution for Problem C is known when $k = 0, 1$

$$\overline{W}^0 = \{id\}; \quad \overline{W}^1 = \{\sigma_j \mid j \in J\},$$

where id is the identity of W and where J is the complement of K in $[1, \cdots, n]$. In general, Algorithm A enables one to build up $\overline{W}^k$ from $\overline{W}^{k-1}$.

Algorithm A. *Decompositions.*

Input. The set $\overline{W}^{k-1}$ being presented by certain reduced decompositions of its elements.

Output. The set $\overline{W}^k$ being presented by certain reduced decompositions of its elements.

Procedure: Set $V = \{1, \cdots, n\} \times \overline{W}^{k-1}$. Repeat the following steps for all elements in V in accordance with the order on V (cf. (4.5)). Begin with empty sets $S = \emptyset$, $R = \emptyset$.

Step 1. For a $v = (i, \sigma_{i_1} \circ \cdots \circ \sigma_{i_{k-1}}) \in V$ form the product $w = \sigma_i \circ \sigma_{i_1} \circ \cdots \circ \sigma_{i_{k-1}}$.

Step 2. Call Algorithm 1 to obtain

$$b(w) = (b_1, \cdots, b_n) \text{ and } b(w^{-1}) = (\overline{b}_1, \cdots, \overline{b}_n);$$

Step 3. If 1) $b_i < 0$;
2) $\overline{b}_i > 0$ for all $i \in K$;
3) $(b_1, \cdots, b_n) \notin R$,

add $\sigma_i \circ \sigma_{i_1} \circ \cdots \circ \sigma_{i_{k-1}}$ to S; add $b(w) = (b_1, \cdots, b_n)$ to R;

The program terminates at $S = \overline{W}^k$.

Explanation. We verify the last clause in Algorithm A. Firstly, Lemma 7 in [DZZ] claims that any $w \in \overline{W}^k$ admits a decomposition $w = \sigma_i \circ \sigma_{i_1} \circ \cdots \circ \sigma_{i_{k-1}}$ for some $(i, \sigma_{i_1} \circ \cdots \circ \sigma_{i_{k-1}}) \in V$. This explains the role the set V plays in the algorithm. Next, the first two conditions in Step 3 guarantees that $\sigma_i \circ \sigma_{i_1} \circ \cdots \circ \sigma_{i_{k-1}} \in \overline{W}^k$ by Lemma 4. Finally, the third constraint in Step 3 rejects a second reduced decomposition of some $w \in \overline{W}^k$ being included in $\overline{W}^k$ (by Lemma 3).□

Remark 2. If $K = \emptyset$, then $\overline{W} = W$ (the whole group). In this case $H = T$ (a maximal torus in G) and Step 2 and 3 in Algorithm A can be simplified as

Step 2. Call Algorithm 1 to obtain $b(w) = (b_1, \cdots, b_n)$;

Step 3. If $b_i < 0$ and if $(b_1, \cdots, b_n) \notin R$, add $\sigma_i \circ \sigma_{i_1} \circ \cdots \circ \sigma_{i_{k-1}}$ to S; add $b(w) = (b_1, \cdots, b_n)$ to R.

Remark 3. Based on the word representation of Weyl groups, a different program solving Problem C was given in [DZZ]. In comparison, the use of the numerical representation simplifies the presentation of Algorithm A.

5 Algorithm B

Algorithm A presents us the coset $\overline{W} = \coprod_{k \geq 0} \overline{W}^k$ by certain reduced decomposition of its elements. Based on this we explain *L-R coefficients*, the algorithm computing $a_{u,v}^w$.

By the notion $L \subset [1, \cdots, k], |L| = r$, we mean that L is a sequence $(j_1, \cdots, j_r)$ of r integers satisfying

$$1 \leq j_1 < \cdots < j_r \leq k.$$

For two integers $1 \leq r \leq k$ let the set $V(k,r) = \{L \mid L \subset [1, \cdots, k], |L| = r\}$ be equipped with the obvious ordering (cf. (4.3)).

For a $w = \sigma_{i_1} \circ \cdots \circ \sigma_{i_k} \in \overline{W}$ and a $u \in \overline{W}$ with $l(u) = r < k$, we set

$$(5.1) \qquad p_w(u) = \sum_{L \in V(k,r), \sigma_L = u} x_L \in \mathbb{Z}[x_1, \cdots, x_k]^{(r)},$$

where $\sigma_L = \sigma_{i_{j_1}} \circ \cdots \circ \sigma_{i_{j_r}}$ if $L = [j_1, \cdots, j_r]$. Using these notations our formula (cf. Section 2) can be simplified as

$$(5.2) \qquad a_{u,v}^w = T_{A_w}[p_w(u) p_w(v)].$$

We begin by pointing out that (5.1) suggests the following algorithm specifying the polynomial $p_w(u)$.

Algorithm 2. Computing $p_w(u) \in \mathbb{Z}[x_1, \cdots, x_k]^{(r)}$.

Input: $w = \sigma_{i_1} \circ \cdots \circ \sigma_{i_k} \in \overline{W}^k$ and $u \in \overline{W}^r$ with $b(u) = (b_1, \cdots, b_n)$.

Output: $p_w(u)$.

Procedure: Repeat the following steps for all $L \in V(k,r)$ in accordance with the order on $V(k,r)$. Initiate the polynomial $p = p(x_1, \cdots, x_k)$ as zero.

Step 1. For a $L \in V(k,r)$ call algorithm 1 to get $b(\sigma_L)$;

Step 2. If $b(\sigma_L) = b(u)$ add x_L to p.

The program terminates at $p = p_w(u)$.

If $A = (a_{ij})_{k\times k}$ is matrix of rank k and if $1 \leq r \leq k-1$, then the notion $(a_{ij})_{r\times r}$ clearly stands for the matrix of rank r obtained from A by deleting the last $(k-r)$ rows and columns.

Let $A = (a_{ij})_{k\times k}$ be a strictly upper triangular integral matrix of rank k. Consider the triangular operator $T_A : \mathbb{Z}[x_1, \cdots, x_k]^{(k)} \to \mathbb{Z}$ given in Definition 2.

Algorithm 3. Computing $T_A : \mathbb{Z}[x_1, \cdots, x_k]^{(k)} \to \mathbb{Z}$.

Input: A strictly upper triangular integral matrix $A = (a_{ij})_{k\times k}$ and a polynomial $p = p(x_1, \cdots, x_k) \in \mathbb{Z}[x_1, \cdots, x_k]^{(k)}$

Output: $T_A(p) \in \mathbb{Z}$.

Procedure: Recursion.

Step 1. Express p as a polynomial in x_k; i.e.

$$p = h_0 + h_1 x_k + \sum_{2\leq r\leq k} h_r x_k^r,\ h_r \in \mathbb{Z}[x_1, \cdots, x_{k-1}]^{(k-r)},$$

and set

$$p_1 = h_1 + \sum_{2\leq r\leq k} h_r (a_{1,k}x_1 + \cdots + a_{k-1,k}x_{k-1})^{r-1} (\in \mathbb{Z}[x_1, \cdots, x_{k-1}]^{(k-1)}).$$

Step 2. Repeat step 1 for $A_1 = (a_{ij})_{(k-1)\times(k-1)}$ and $p = p_1$ to get $p_2 \in \mathbb{Z}[x_1, \cdots, x_{k-2}]^{(k-2)}$.

$\vdots$

Step k+1. If $p_k = a \in \mathbb{Z}$, then $T_A(p) = a$.

Algorithm B. *L-R coefficients.*

Input: $w = \sigma_{i_1} \circ \cdots \circ \sigma_{i_k} \in \overline{W}^k$, $(u, v) \in \overline{W}^r \times \overline{W}^{k-r}$

Output: $a_{u,v}^w \in \mathbb{Z}$.

Procedure: Let A_w be the Cartan matrix of w related to the decomposition (it can be read directly from the Cartan matrix of G and the decomposition $w = \sigma_{i_1} \circ \cdots \circ \sigma_{i_k}$. cf. Definition 1).

Step 1. Call algorithm 2 to get $p_w(u)$ and $p_w(v)$;

Step 2. Call algorithm 3 to get $T_{A_w}(p_w(u) \cdot p_w(v))$.

Step 3. If $T_{A_w}(p_w(u) \cdot p_w(v)) = a$, then $a_{u,v}^w = a$ (by (5.2)).

Remark 4. Based on Algorithm B, a parallel program to expand the product

$$P_u \cdot P_v = \sum_{w\in \overline{W}^k} a_{u,v}^w P_w$$

for given $(u, v) \in \overline{W}^r \times \overline{W}^{k-r}$ can be easily implemented. The order on $\overline{W}^k$ can be employed to assign each $w \in \overline{W}^k$ a computing unit.

6 Computational examples

Our algorithm is ready to apply to computation in flag manifolds. Recall that all compact connected semi-simple irreducible Lie groups fall into four infinite sequences of matrix groups

$$SU(n); \quad SO(2n); \quad SO(2n+1); \quad Sp(n),$$

as well as the five exceptional ones

$$G_2, \qquad F_4, \qquad E_6, \qquad E_7, \qquad E_8.$$

The flag manifolds associated to matrix groups have been studied extensively during the past decades. Here, we choose to work with certain flag manifolds related to the exceptional Lie groups E_n, $n = 6, 7, 8$.

Fix a maximal torus $T^n \subset E_n$ and let $W(n)$ be the Weyl group of E_n. Then

$$(6.1) \qquad |W(n)| = \begin{cases} 2^7 3^4 5 \text{ if } n = 6; \\ 2^{10} 3^4 5 7 \text{ if } n = 7; \\ 2^{14} 3^5 5^2 7 \text{ if } n = 8; \end{cases} \qquad \dim_{\mathbb{R}} E_n = \begin{cases} 78 \text{ if } n = 6; \\ 133 \text{ if } n = 7; \\ 248 \text{ if } n = 8, \end{cases}$$

where $|A|$ stands for the cardinality of the set A. Assume that the set of simple roots $\Delta = \{\beta_1, \cdots, \beta_n\}$ of E_n is given and ordered as the vertices of the Dynkin diagram of E_n pictured in [Hu, p.58], and let $K \subset \{1, 2, \cdots, n\}$ be the subset whose complement is $\{2\}$. We have the following information on the subgroup group H_K.

(a) *the semisimple part of the subgroup $H_K \subset E_n$ is $SU(n)$, the special unitary group of order n;*

(b) *H_K admits a factorization into the semi-product $H_K = S^1 \cdot SU(n)$, where S^1 is a circle subgroup of the maximal torus T^n in E_n;*

(c) *if $W'(n) \subset W(n)$ is the Weyl group of H_K, then $|W'(n)| = n!$.*

Consequently, if one write $\overline{W}(n)$ for the coset $W^{'}(n)$ in $W(n)$, one has

$$(6.2) \qquad \left|\overline{W}(n)\right| = \begin{cases} 2^3 3^2 \text{ if } n = 6; \\ 2^6 3^2 \text{ if } n = 7; \\ 2^7 3^3 5 \text{ if } n = 8. \end{cases} \qquad \dim_{\mathbb{R}} E_n/H_K = \begin{cases} 42 \text{ if } n = 6; \\ 84 \text{ if } n = 7; \\ 194 \text{ if } n = 8. \end{cases}$$

Geometrically, $\overline{W}(n)$ parameterizes Schubert classes on E_n/H_K (i.e. the Basis Theorem).

The subset of $\overline{W}(n)$ consisting of elements with length r is denoted $\overline{W}^r(n)$. By (4.4), if the $\overline{W}^r(n)$ is presented by its elements each with a reduced decompositions, then it naturally becomes an ordered set and therefore, can be alternatively presented as

$$(6.3) \qquad \overline{W}^r(n) = \{w_{r,i} \mid 1 \le i \le |\overline{W}^r|\}.$$

In table A_n below, we present elements of $\overline{W}(n)$ with length $r \le 10$ both in terms of their reduced decompositions produced by Algorithm A, and the index system (6.3) imposed by the decompositions.

The index (6.3) on $\overline{W}^r(n)$ is useful in simplifying the presentation of the intersection multiplicities $a^w_{u,v}$. By resorting to this index system we list in table B_n ($n = 6, 7, 8$) all the $a^w_{u,v}$ with $l(w) = 9$ and 10 produced by Algorithm B.

Table $\mathbf{A}_6$. *Reduced decomposition of elements in* $\overline{W}(6)$ *with length* ≤ 10

$w_{i,j}$	decomposition		$w_{i,j}$	decomposition	
$w_{1,1}$	σ_2		$w_{7,5}$	$\sigma_5\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$	
$w_{2,1}$	$\sigma_4\sigma_2$		$w_{8,1}$	$\sigma_1\sigma_2\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$	
$w_{3,1}$	$\sigma_3\sigma_4\sigma_2$		$w_{8,2}$	$\sigma_1\sigma_5\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$	
$w_{3,2}$	$\sigma_5\sigma_4\sigma_2$		$w_{8,3}$	$\sigma_2\sigma_3\sigma_1\sigma_4\sigma_3\sigma_5\sigma_4\sigma_2$	
$w_{4,1}$	$\sigma_1\sigma_3\sigma_4\sigma_2$		$w_{8,4}$	$\sigma_2\sigma_5\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$	
$w_{4,2}$	$\sigma_3\sigma_5\sigma_4\sigma_2$		$w_{8,5}$	$\sigma_3\sigma_1\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$	
$w_{4,3}$	$\sigma_6\sigma_5\sigma_4\sigma_2$		$w_{9,1}$	$\sigma_1\sigma_2\sigma_5\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$	
$w_{5,1}$	$\sigma_1\sigma_3\sigma_5\sigma_4\sigma_2$		$w_{9,2}$	$\sigma_2\sigma_3\sigma_1\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$	
$w_{5,2}$	$\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$		$w_{9,3}$	$\sigma_3\sigma_1\sigma_5\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$	
$w_{5,3}$	$\sigma_4\sigma_3\sigma_5\sigma_4\sigma_2$		$w_{9,4}$	$\sigma_4\sigma_2\sigma_3\sigma_1\sigma_4\sigma_3\sigma_5\sigma_4\sigma_2$	
$w_{6,1}$	$\sigma_1\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$		$w_{9,5}$	$\sigma_4\sigma_2\sigma_5\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$	
$w_{6,2}$	$\sigma_1\sigma_4\sigma_3\sigma_5\sigma_4\sigma_2$		$w_{10,1}$	$\sigma_1\sigma_4\sigma_2\sigma_5\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$	
$w_{6,3}$	$\sigma_2\sigma_4\sigma_3\sigma_5\sigma_4\sigma_2$		$w_{10,2}$	$\sigma_2\sigma_3\sigma_1\sigma_5\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$	
$w_{6,4}$	$\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$		$w_{10,3}$	$\sigma_3\sigma_4\sigma_2\sigma_5\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$	
$w_{7,1}$	$\sigma_1\sigma_2\sigma_4\sigma_3\sigma_5\sigma_4\sigma_2$		$w_{10,4}$	$\sigma_4\sigma_2\sigma_3\sigma_1\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$	
$w_{7,2}$	$\sigma_1\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$		$w_{10,5}$	$\sigma_4\sigma_3\sigma_1\sigma_5\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$	
$w_{7,3}$	$\sigma_2\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$		$w_{10,6}$	$\sigma_5\sigma_4\sigma_2\sigma_3\sigma_1\sigma_4\sigma_3\sigma_5\sigma_4\sigma_2$	
$w_{7,4}$	$\sigma_3\sigma_1\sigma_4\sigma_3\sigma_5\sigma_4\sigma_2$				

Table $\mathbf{A}_7$. *Reduced decomposition of elements in* $\overline{W}(7)$ *with length* ≤ 10

$w_{i,j}$	decomposition	$w_{i,j}$	decomposition	$w_{i,j}$	decomposition
$w_{1,1}$	σ_2	$w_{7,3}$	$\sigma_1\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{9,6}$	$\sigma_3\sigma_1\sigma_4\sigma_3\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$
$w_{2,1}$	$\sigma_4\sigma_2$	$w_{7,4}$	$\sigma_2\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{9,7}$	$\sigma_3\sigma_1\sigma_5\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$
$w_{3,1}$	$\sigma_3\sigma_4\sigma_2$	$w_{7,5}$	$\sigma_3\sigma_1\sigma_4\sigma_3\sigma_5\sigma_4\sigma_2$	$w_{9,8}$	$\sigma_4\sigma_2\sigma_3\sigma_1\sigma_4\sigma_3\sigma_5\sigma_4\sigma_2$
$w_{3,2}$	$\sigma_5\sigma_4\sigma_2$	$w_{7,6}$	$\sigma_4\sigma_3\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{9,9}$	$\sigma_4\sigma_2\sigma_5\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$
$w_{4,1}$	$\sigma_1\sigma_3\sigma_4\sigma_2$	$w_{7,7}$	$\sigma_5\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{9,10}$	$\sigma_6\sigma_5\sigma_4\sigma_3\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$
$w_{4,2}$	$\sigma_3\sigma_5\sigma_4\sigma_2$	$w_{8,1}$	$\sigma_1\sigma_2\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{10,1}$	$\sigma_1\sigma_2\sigma_5\sigma_4\sigma_3\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$
$w_{4,3}$	$\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{8,2}$	$\sigma_1\sigma_4\sigma_3\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{10,2}$	$\sigma_1\sigma_4\sigma_2\sigma_5\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$
$w_{5,1}$	$\sigma_1\sigma_3\sigma_5\sigma_4\sigma_2$	$w_{8,3}$	$\sigma_1\sigma_5\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{10,3}$	$\sigma_1\sigma_6\sigma_5\sigma_4\sigma_3\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$
$w_{5,2}$	$\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{8,4}$	$\sigma_2\sigma_3\sigma_1\sigma_4\sigma_3\sigma_5\sigma_4\sigma_2$	$w_{10,4}$	$\sigma_2\sigma_3\sigma_1\sigma_4\sigma_3\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$
$w_{5,3}$	$\sigma_4\sigma_3\sigma_5\sigma_4\sigma_2$	$w_{8,5}$	$\sigma_2\sigma_4\sigma_3\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{10,5}$	$\sigma_2\sigma_3\sigma_1\sigma_5\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$
$w_{5,4}$	$\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{8,6}$	$\sigma_2\sigma_5\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{10,6}$	$\sigma_2\sigma_6\sigma_5\sigma_4\sigma_3\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$
$w_{6,1}$	$\sigma_1\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{8,7}$	$\sigma_3\sigma_1\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{10,7}$	$\sigma_3\sigma_1\sigma_5\sigma_4\sigma_3\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$
$w_{6,2}$	$\sigma_1\sigma_4\sigma_3\sigma_5\sigma_4\sigma_2$	$w_{8,8}$	$\sigma_5\sigma_4\sigma_3\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{10,8}$	$\sigma_3\sigma_4\sigma_2\sigma_5\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$
$w_{6,3}$	$\sigma_2\sigma_4\sigma_3\sigma_5\sigma_4\sigma_2$	$w_{9,1}$	$\sigma_1\sigma_2\sigma_4\sigma_3\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{10,9}$	$\sigma_4\sigma_2\sigma_3\sigma_1\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$
$w_{6,4}$	$\sigma_3\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{9,2}$	$\sigma_1\sigma_2\sigma_5\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{10,10}$	$\sigma_4\sigma_2\sigma_5\sigma_4\sigma_3\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$
$w_{6,5}$	$\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{9,3}$	$\sigma_1\sigma_5\sigma_4\sigma_3\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{10,11}$	$\sigma_4\sigma_3\sigma_1\sigma_5\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$
$w_{7,1}$	$\sigma_1\sigma_2\sigma_4\sigma_3\sigma_5\sigma_4\sigma_2$	$w_{9,4}$	$\sigma_2\sigma_3\sigma_1\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{10,12}$	$\sigma_5\sigma_4\sigma_2\sigma_3\sigma_1\sigma_4\sigma_3\sigma_5\sigma_4\sigma_2$
$w_{7,2}$	$\sigma_1\sigma_3\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{9,5}$	$\sigma_2\sigma_5\sigma_4\sigma_3\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$		

Table $\mathbf{A}_8$.*Reduced decomposition of elements in* $\overline{W}(8)$ *with length* ≤ 10.

$w_{i,j}$	decomposition	$w_{i,j}$	decomposition	$w_{i,j}$	decomposition
$w_{1,1}$	σ_2	$w_{7,6}$	$\sigma_3\sigma_8\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{9,10}$	$\sigma_4\sigma_2\sigma_3\sigma_1\sigma_4\sigma_3\sigma_5\sigma_4\sigma_2$
$w_{2,1}$	$\sigma_4\sigma_2$	$w_{7,7}$	$\sigma_4\sigma_3\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{9,11}$	$\sigma_4\sigma_2\sigma_5\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$
$w_{3,1}$	$\sigma_3\sigma_4\sigma_2$	$w_{7,8}$	$\sigma_5\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{9,12}$	$\sigma_5\sigma_4\sigma_3\sigma_8\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$
$w_{3,2}$	$\sigma_5\sigma_4\sigma_2$	$w_{8,1}$	$\sigma_1\sigma_2\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{9,13}$	$\sigma_6\sigma_5\sigma_4\sigma_3\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$
$w_{4,1}$	$\sigma_1\sigma_3\sigma_4\sigma_2$	$w_{8,2}$	$\sigma_1\sigma_3\sigma_8\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{10,1}$	$\sigma_1\sigma_2\sigma_4\sigma_3\sigma_8\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$
$w_{4,2}$	$\sigma_3\sigma_5\sigma_4\sigma_2$	$w_{8,3}$	$\sigma_1\sigma_4\sigma_3\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{10,2}$	$\sigma_1\sigma_2\sigma_5\sigma_4\sigma_3\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$
$w_{4,3}$	$\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{8,4}$	$\sigma_1\sigma_5\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{10,3}$	$\sigma_1\sigma_4\sigma_2\sigma_5\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$
$w_{5,1}$	$\sigma_1\sigma_3\sigma_5\sigma_4\sigma_2$	$w_{8,5}$	$\sigma_2\sigma_3\sigma_1\sigma_4\sigma_3\sigma_5\sigma_4\sigma_2$	$w_{10,4}$	$\sigma_1\sigma_5\sigma_4\sigma_3\sigma_8\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$
$w_{5,2}$	$\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{8,6}$	$\sigma_2\sigma_4\sigma_3\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{10,5}$	$\sigma_1\sigma_6\sigma_5\sigma_4\sigma_3\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$
$w_{5,3}$	$\sigma_4\sigma_3\sigma_5\sigma_4\sigma_2$	$w_{8,7}$	$\sigma_2\sigma_5\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{10,6}$	$\sigma_2\sigma_3\sigma_1\sigma_4\sigma_3\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$
$w_{5,4}$	$\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{8,8}$	$\sigma_3\sigma_1\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{10,7}$	$\sigma_2\sigma_3\sigma_1\sigma_5\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$
$w_{6,1}$	$\sigma_1\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{8,9}$	$\sigma_4\sigma_3\sigma_8\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{10,8}$	$\sigma_2\sigma_5\sigma_4\sigma_3\sigma_8\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$
$w_{6,2}$	$\sigma_1\sigma_4\sigma_3\sigma_5\sigma_4\sigma_2$	$w_{8,10}$	$\sigma_5\sigma_4\sigma_3\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{10,9}$	$\sigma_2\sigma_6\sigma_5\sigma_4\sigma_3\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$
$w_{6,3}$	$\sigma_2\sigma_4\sigma_3\sigma_5\sigma_4\sigma_2$	$w_{9,1}$	$\sigma_1\sigma_2\sigma_4\sigma_3\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{10,10}$	$\sigma_3\sigma_1\sigma_4\sigma_3\sigma_8\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$
$w_{6,4}$	$\sigma_3\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{9,2}$	$\sigma_1\sigma_2\sigma_5\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{10,11}$	$\sigma_3\sigma_1\sigma_5\sigma_4\sigma_3\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$
$w_{6,5}$	$\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{9,3}$	$\sigma_1\sigma_4\sigma_3\sigma_8\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{10,12}$	$\sigma_3\sigma_4\sigma_2\sigma_5\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$
$w_{6,6}$	$\sigma_8\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{9,4}$	$\sigma_1\sigma_5\sigma_4\sigma_3\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{10,13}$	$\sigma_4\sigma_2\sigma_3\sigma_1\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$
$w_{7,1}$	$\sigma_1\sigma_2\sigma_4\sigma_3\sigma_5\sigma_4\sigma_2$	$w_{9,5}$	$\sigma_2\sigma_3\sigma_1\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{10,14}$	$\sigma_4\sigma_2\sigma_5\sigma_4\sigma_3\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$
$w_{7,2}$	$\sigma_1\sigma_3\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{9,6}$	$\sigma_2\sigma_4\sigma_3\sigma_8\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{10,15}$	$\sigma_4\sigma_3\sigma_1\sigma_5\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$
$w_{7,3}$	$\sigma_1\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{9,7}$	$\sigma_2\sigma_5\sigma_4\sigma_3\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{10,16}$	$\sigma_5\sigma_4\sigma_2\sigma_3\sigma_1\sigma_4\sigma_3\sigma_5\sigma_4\sigma_2$
$w_{7,4}$	$\sigma_2\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{9,8}$	$\sigma_3\sigma_1\sigma_4\sigma_3\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$	$w_{10,17}$	$\sigma_6\sigma_5\sigma_4\sigma_3\sigma_8\sigma_7\sigma_6\sigma_5\sigma_4\sigma_2$
$w_{7,5}$	$\sigma_3\sigma_1\sigma_4\sigma_3\sigma_5\sigma_4\sigma_2$	$w_{9,9}$	$\sigma_3\sigma_1\sigma_5\sigma_4\sigma_3\sigma_6\sigma_5\sigma_4\sigma_2$		

Table B$_6$. *L-R coefficients for* $E_6/S^1 \cdot SU(6)$

u	v	$w \in W^9(6)$				
		$w_{9,1}$	$w_{9,2}$	$w_{9,3}$	$w_{9,4}$	$w_{9,5}$
$w_{1,1}$	$w_{8,1}$	1	1	0	0	0
$w_{1,1}$	$w_{8,2}$	1	0	1	0	0
$w_{1,1}$	$w_{8,3}$	0	1	0	1	0
$w_{1,1}$	$w_{8,4}$	1	0	0	0	1
$w_{1,1}$	$w_{8,5}$	0	1	1	0	0
$w_{2,1}$	$w_{7,1}$	1	2	0	1	0
$w_{2,1}$	$w_{7,2}$	2	2	2	0	0
$w_{2,1}$	$w_{7,3}$	2	1	0	0	1
$w_{2,1}$	$w_{7,4}$	0	2	1	1	0
$w_{2,1}$	$w_{7,5}$	2	0	1	0	1
$w_{3,1}$	$w_{6,1}$	1	1	1	0	0
$w_{3,1}$	$w_{6,2}$	1	3	2	1	0
$w_{3,1}$	$w_{6,3}$	2	1	0	1	0
$w_{3,1}$	$w_{6,4}$	3	2	1	0	1
$w_{3,2}$	$w_{6,1}$	1	1	1	0	0
$w_{3,2}$	$w_{6,2}$	2	3	1	1	0
$w_{3,2}$	$w_{6,3}$	1	2	0	0	1
$w_{3,2}$	$w_{6,4}$	3	1	2	0	1
$w_{4,1}$	$w_{5,1}$	0	1	1	0	0
$w_{4,1}$	$w_{5,2}$	1	1	0	0	0
$w_{4,1}$	$w_{5,3}$	1	1	1	1	0
$w_{4,2}$	$w_{5,1}$	2	3	2	1	0
$w_{4,2}$	$w_{5,2}$	3	2	2	0	1
$w_{4,2}$	$w_{5,3}$	5	5	2	1	1
$w_{4,3}$	$w_{5,1}$	1	1	0	0	0
$w_{4,3}$	$w_{5,2}$	1	0	1	0	0
$w_{4,3}$	$w_{5,3}$	1	1	1	0	1

u	v	$w \in W^{10}(6)$					
		$w_{10,1}$	$w_{10,2}$	$w_{10,3}$	$w_{10,4}$	$w_{10,5}$	$w_{10,6}$
$w_{1,1}$	$w_{9,1}$	1	1	0	0	0	0
$w_{1,1}$	$w_{9,2}$	0	1	0	1	0	0
$w_{1,1}$	$w_{9,3}$	0	1	0	0	1	0
$w_{1,1}$	$w_{9,4}$	0	0	0	1	0	1
$w_{1,1}$	$w_{9,5}$	1	0	1	0	0	0
$w_{2,1}$	$w_{8,1}$	1	2	0	1	0	0
$w_{2,1}$	$w_{8,2}$	1	2	0	0	1	0
$w_{2,1}$	$w_{8,3}$	0	1	0	2	0	1
$w_{2,1}$	$w_{8,4}$	2	1	1	0	0	0
$w_{2,1}$	$w_{8,5}$	0	2	0	1	1	0
$w_{3,1}$	$w_{7,1}$	0	2	0	1	0	1
$w_{3,1}$	$w_{7,2}$	1	3	0	1	1	0
$w_{3,1}$	$w_{7,3}$	2	1	0	1	0	0
$w_{3,1}$	$w_{7,4}$	0	1	0	2	1	0
$w_{3,1}$	$w_{7,5}$	1	2	1	0	0	0
$w_{3,2}$	$w_{7,1}$	1	1	0	2	0	0
$w_{3,2}$	$w_{7,2}$	1	3	0	1	1	0
$w_{3,2}$	$w_{7,3}$	1	2	1	0	0	0
$w_{3,2}$	$w_{7,4}$	0	2	0	1	0	1
$w_{3,2}$	$w_{7,5}$	2	1	0	0	1	0
$w_{4,1}$	$w_{6,1}$	0	1	0	0	0	0
$w_{4,1}$	$w_{6,2}$	0	1	0	1	1	0
$w_{4,1}$	$w_{6,3}$	0	1	0	0	0	1
$w_{4,1}$	$w_{6,4}$	1	1	0	1	0	0
$w_{4,2}$	$w_{6,1}$	1	2	0	1	1	0
$w_{4,2}$	$w_{6,2}$	1	5	0	3	1	1
$w_{4,2}$	$w_{6,3}$	2	2	0	2	0	0
$w_{4,2}$	$w_{6,4}$	3	5	1	1	1	0
$w_{4,3}$	$w_{6,1}$	0	1	0	0	0	0
$w_{4,3}$	$w_{6,2}$	1	1	0	1	0	0
$w_{4,3}$	$w_{6,3}$	0	1	1	0	0	0
$w_{4,3}$	$w_{6,4}$	1	1	0	0	1	0
$w_{5,1}$	$w_{5,1}$	0	2	0	1	1	0
$w_{5,1}$	$w_{5,2}$	1	2	0	1	0	0
$w_{5,1}$	$w_{5,3}$	1	3	0	2	1	1
$w_{5,2}$	$w_{5,2}$	1	2	0	0	1	0
$w_{5,2}$	$w_{5,3}$	2	3	1	1	1	0
$w_{5,3}$	$w_{5,3}$	3	6	0	3	0	0

Table $\mathbf{B}_7$. *L-R coefficients for* $E_7/S^1 \cdot SU(7)$

u	v	$w \in W^9(7)$									
		$w_{9,1}$	$w_{9,2}$	$w_{9,3}$	$w_{9,4}$	$w_{9,5}$	$w_{9,6}$	$w_{9,7}$	$w_{9,8}$	$w_{9,9}$	$w_{9,10}$
$w_{1,1}$	$w_{8,1}$	1	1	0	1	0	0	0	0	0	0
$w_{1,1}$	$w_{8,2}$	1	0	1	0	0	1	0	0	0	0
$w_{1,1}$	$w_{8,3}$	0	1	1	0	0	0	1	0	0	0
$w_{1,1}$	$w_{8,4}$	0	0	0	1	0	0	0	1	0	0
$w_{1,1}$	$w_{8,5}$	1	0	0	0	1	0	0	0	0	0
$w_{1,1}$	$w_{8,6}$	0	1	0	0	1	0	0	0	1	0
$w_{1,1}$	$w_{8,7}$	0	0	0	1	0	1	1	0	0	0
$w_{1,1}$	$w_{8,8}$	0	0	1	0	1	0	0	0	0	1
$w_{2,1}$	$w_{7,1}$	1	1	0	2	0	0	0	1	0	0
$w_{2,1}$	$w_{7,2}$	1	0	1	0	0	1	0	0	0	0
$w_{2,1}$	$w_{7,3}$	2	2	2	2	0	2	2	0	0	0
$w_{2,1}$	$w_{7,4}$	2	2	0	1	2	0	0	0	1	0
$w_{2,1}$	$w_{7,5}$	0	0	0	2	0	1	1	1	0	0
$w_{2,1}$	$w_{7,6}$	2	0	2	0	2	1	0	0	0	1
$w_{2,1}$	$w_{7,7}$	0	2	2	0	2	0	1	0	1	1
$w_{3,1}$	$w_{6,1}$	1	1	1	1	0	2	1	0	0	0
$w_{3,1}$	$w_{6,2}$	1	1	1	3	0	1	2	1	0	0
$w_{3,1}$	$w_{6,3}$	1	2	0	1	1	0	0	1	0	0
$w_{3,1}$	$w_{6,4}$	2	0	1	0	1	1	0	0	0	0
$w_{3,1}$	$w_{6,5}$	3	3	3	2	2	1	1	0	1	1
$w_{3,2}$	$w_{6,1}$	2	1	2	1	0	1	1	0	0	0
$w_{3,2}$	$w_{6,2}$	2	2	1	3	0	2	1	1	0	0
$w_{3,2}$	$w_{6,3}$	2	1	0	2	1	0	0	0	1	0
$w_{3,2}$	$w_{6,4}$	1	0	2	0	1	1	0	0	0	1
$w_{3,2}$	$w_{6,5}$	3	3	3	1	4	2	2	0	1	1
$w_{4,1}$	$w_{5,1}$	0	0	0	1	0	1	1	0	0	0
$w_{4,1}$	$w_{5,2}$	1	1	1	1	0	1	0	0	0	0
$w_{4,1}$	$w_{5,3}$	1	1	1	1	0	0	1	1	0	0
$w_{4,1}$	$w_{5,4}$	1	0	0	0	0	0	0	0	0	0
$w_{4,2}$	$w_{5,1}$	2	2	2	3	0	2	2	1	0	0
$w_{4,2}$	$w_{5,2}$	5	3	4	2	3	3	2	0	1	1
$w_{4,2}$	$w_{5,3}$	4	5	3	5	3	2	2	1	1	1
$w_{4,2}$	$w_{5,4}$	1	0	1	0	1	1	0	0	0	0
$w_{4,3}$	$w_{5,1}$	2	1	1	1	0	1	0	0	0	0
$w_{4,3}$	$w_{5,2}$	1	1	3	0	2	1	1	0	0	1
$w_{4,3}$	$w_{5,3}$	3	1	1	1	2	2	1	0	1	0
$w_{4,3}$	$w_{5,4}$	0	0	1	0	0	0	0	0	0	1

u	v	$w \in W^{10}(7)$											
		$w_{10,1}$	$w_{10,2}$	$w_{10,3}$	$w_{10,4}$	$w_{10,5}$	$w_{10,6}$	$w_{10,7}$	$w_{10,8}$	$w_{10,9}$	$w_{10,10}$	$w_{10,11}$	$w_{10,12}$
$w_{1,1}$	$w_{9,1}$	1	0	0	1	0	0	0	0	0	0	0	0
$w_{1,1}$	$w_{9,2}$	1	1	0	0	1	0	0	0	0	0	0	0
$w_{1,1}$	$w_{9,3}$	1	0	1	0	0	0	1	0	0	0	0	0
$w_{1,1}$	$w_{9,4}$	0	0	0	1	1	0	0	0	1	0	0	0
$w_{1,1}$	$w_{9,5}$	1	0	0	0	0	1	0	0	0	1	0	0
$w_{1,1}$	$w_{9,6}$	0	0	0	1	0	0	1	0	0	0	0	0
$w_{1,1}$	$w_{9,7}$	0	0	0	0	1	0	1	0	0	0	1	0
$w_{1,1}$	$w_{9,8}$	0	0	0	0	0	0	0	0	1	0	0	1
$w_{1,1}$	$w_{9,9}$	0	1	0	0	0	0	0	1	0	1	0	0
$w_{1,1}$	$w_{9,10}$	0	0	1	0	0	1	0	0	0	0	0	0
$w_{2,1}$	$w_{8,1}$	2	1	0	2	2	0	0	0	1	0	0	0
$w_{2,1}$	$w_{8,2}$	2	0	1	2	0	0	2	0	0	0	0	0
$w_{2,1}$	$w_{8,3}$	2	1	1	0	2	0	2	0	0	0	1	0
$w_{2,1}$	$w_{8,4}$	0	0	0	1	1	0	0	0	2	0	0	1
$w_{2,1}$	$w_{8,5}$	2	0	0	1	0	1	0	0	0	1	0	0
$w_{2,1}$	$w_{8,6}$	2	2	0	0	1	1	0	1	0	2	0	0
$w_{2,1}$	$w_{8,7}$	0	0	0	2	2	0	2	0	1	0	1	0
$w_{2,1}$	$w_{8,8}$	2	0	2	0	0	2	1	0	0	1	0	0
$w_{3,1}$	$w_{7,1}$	1	0	0	1	2	0	0	0	1	0	0	1
$w_{3,1}$	$w_{7,2}$	1	0	0	1	0	0	1	0	0	0	0	0
$w_{3,1}$	$w_{7,3}$	2	1	1	3	3	0	3	0	1	0	1	0
$w_{3,1}$	$w_{7,4}$	3	2	0	1	1	1	0	0	1	1	0	0
$w_{3,1}$	$w_{7,5}$	0	0	0	1	1	0	1	0	2	0	1	0
$w_{3,1}$	$w_{7,6}$	3	0	1	2	0	1	1	0	0	1	0	0
$w_{3,1}$	$w_{7,7}$	3	1	2	0	2	1	1	1	0	1	0	0
$w_{3,2}$	$w_{7,1}$	1	1	0	2	1	0	0	0	2	0	0	0
$w_{3,2}$	$w_{7,2}$	1	0	1	1	0	0	1	0	0	0	0	0
$w_{3,2}$	$w_{7,3}$	4	1	1	3	3	0	3	0	1	0	1	0
$w_{3,2}$	$w_{7,4}$	3	1	0	2	2	1	0	1	0	2	0	0
$w_{3,2}$	$w_{7,5}$	0	0	0	2	2	0	1	0	1	0	0	1
$w_{3,2}$	$w_{7,6}$	3	0	2	1	0	2	2	0	0	1	0	0
$w_{3,2}$	$w_{7,7}$	3	2	1	0	1	2	2	0	0	2	1	0
$w_{4,1}$	$w_{6,1}$	0	0	0	1	1	0	1	0	0	0	0	0
$w_{4,1}$	$w_{6,2}$	0	0	0	1	1	0	1	0	1	0	1	0
$w_{4,1}$	$w_{6,3}$	1	0	0	0	1	0	0	0	0	0	0	1
$w_{4,1}$	$w_{6,4}$	1	0	0	1	0	0	0	0	0	0	0	0
$w_{4,1}$	$w_{6,5}$	2	1	1	1	1	0	1	0	1	0	0	0
$w_{4,2}$	$w_{6,1}$	3	1	1	3	2	0	3	0	1	0	1	0
$w_{4,2}$	$w_{6,2}$	3	1	1	4	5	0	3	0	3	0	1	1
$w_{4,2}$	$w_{6,3}$	3	2	0	2	2	1	0	0	2	1	0	0
$w_{4,2}$	$w_{6,4}$	3	0	1	2	0	1	2	0	0	1	0	0
$w_{4,2}$	$w_{6,5}$	9	3	3	5	5	3	4	1	1	3	1	0
$w_{4,3}$	$w_{6,1}$	2	0	1	1	1	0	1	0	0	0	0	0
$w_{4,3}$	$w_{6,2}$	2	1	0	3	1	0	1	0	1	0	0	0
$w_{4,3}$	$w_{6,3}$	1	0	0	2	1	0	0	1	0	1	0	0
$w_{4,3}$	$w_{6,4}$	1	0	2	0	0	1	1	0	0	0	0	0
$w_{4,3}$	$w_{6,5}$	4	1	1	1	1	2	3	0	0	2	1	0
$w_{5,1}$	$w_{5,1}$	0	0	0	2	2	0	2	0	1	0	1	0
$w_{5,1}$	$w_{5,2}$	3	1	1	3	2	0	2	0	1	0	0	0
$w_{5,1}$	$w_{5,3}$	3	1	1	2	3	0	2	0	2	0	1	1
$w_{5,1}$	$w_{5,4}$	1	0	0	1	0	0	0	0	0	0	0	0
$w_{5,2}$	$w_{5,2}$	6	1	3	2	2	2	4	0	0	2	1	0
$w_{5,2}$	$w_{5,3}$	6	2	1	5	3	2	3	1	1	2	1	0
$w_{5,2}$	$w_{5,4}$	1	0	1	0	0	1	1	0	0	0	0	0
$w_{5,3}$	$w_{5,3}$	6	3	2	4	6	2	2	0	3	2	0	0
$w_{5,3}$	$w_{5,4}$	1	0	0	1	0	0	1	0	0	1	0	0
$w_{5,4}$	$w_{5,4}$	0	0	1	0	0	0	0	0	0	0	0	0

Table B$_8$. *L-R coefficients for $E_8/S^1 \cdot SU(8)$*

u	v	$w \in W^9(8)$												
		$w_{9,1}$	$w_{9,2}$	$w_{9,3}$	$w_{9,4}$	$w_{9,5}$	$w_{9,6}$	$w_{9,7}$	$w_{9,8}$	$w_{9,9}$	$w_{9,10}$	$w_{9,11}$	$w_{9,12}$	$w_{9,13}$
$w_{1,1}$	$w_{8,1}$	1	1	0	0	1	0	0	0	0	0	0	0	0
$w_{1,1}$	$w_{8,2}$	0	0	1	0	0	0	0	0	0	0	0	0	0
$w_{1,1}$	$w_{8,3}$	1	0	1	1	0	0	0	1	0	0	0	0	0
$w_{1,1}$	$w_{8,4}$	0	1	0	1	0	0	0	0	1	0	0	0	0
$w_{1,1}$	$w_{8,5}$	0	0	0	0	1	0	0	0	0	1	0	0	0
$w_{1,1}$	$w_{8,6}$	1	0	0	0	0	1	1	0	0	0	0	0	0
$w_{1,1}$	$w_{8,7}$	0	1	0	0	0	0	1	0	0	0	1	0	0
$w_{1,1}$	$w_{8,8}$	0	0	0	0	1	0	0	1	1	0	0	0	0
$w_{1,1}$	$w_{8,9}$	0	0	1	0	0	1	0	0	0	0	0	1	0
$w_{1,1}$	$w_{8,10}$	0	0	0	1	0	0	1	0	0	0	0	1	1
$w_{2,1}$	$w_{7,1}$	1	1	0	0	2	0	0	0	0	1	0	0	0
$w_{2,1}$	$w_{7,2}$	1	0	2	1	0	0	0	1	0	0	0	0	0
$w_{2,1}$	$w_{7,3}$	2	2	1	2	2	0	0	2	2	0	0	0	0
$w_{2,1}$	$w_{7,4}$	2	2	0	0	1	1	2	0	0	0	1	0	0
$w_{2,1}$	$w_{7,5}$	0	0	0	0	2	0	0	1	1	1	0	0	0
$w_{2,1}$	$w_{7,6}$	0	0	2	0	0	1	0	0	0	0	0	1	0
$w_{2,1}$	$w_{7,7}$	2	0	2	2	0	2	2	1	0	0	0	2	1
$w_{2,1}$	$w_{7,8}$	0	2	0	2	0	0	2	0	1	0	1	1	1
$w_{3,1}$	$w_{6,1}$	1	1	1	1	1	0	0	2	1	0	0	0	0
$w_{3,1}$	$w_{6,2}$	1	1	0	1	3	0	0	1	2	1	0	0	0
$w_{3,1}$	$w_{6,3}$	1	2	0	0	1	0	1	0	0	1	0	0	0
$w_{3,1}$	$w_{6,4}$	2	0	3	1	0	1	1	1	0	0	0	1	0
$w_{3,1}$	$w_{6,5}$	3	3	1	3	2	1	2	1	1	0	1	1	1
$w_{3,1}$	$w_{6,6}$	0	0	1	0	0	1	0	0	0	0	0	0	0
$w_{3,2}$	$w_{6,1}$	2	1	2	2	1	0	0	1	1	0	0	0	0
$w_{3,2}$	$w_{6,2}$	2	2	1	1	3	0	0	2	1	1	0	0	0
$w_{3,2}$	$w_{6,3}$	2	1	0	0	2	1	1	0	0	0	1	0	0
$w_{3,2}$	$w_{6,4}$	1	0	3	2	0	2	1	1	0	0	0	2	1
$w_{3,2}$	$w_{6,5}$	3	3	2	3	1	2	4	2	2	0	1	2	1
$w_{3,2}$	$w_{6,6}$	0	0	1	0	0	0	0	0	0	0	0	1	0
$w_{4,1}$	$w_{5,1}$	0	0	0	0	1	0	0	1	1	0	0	0	0
$w_{4,1}$	$w_{5,2}$	1	1	1	1	1	0	0	1	0	0	0	0	0
$w_{4,1}$	$w_{5,3}$	1	1	0	1	1	0	0	0	1	1	0	0	0
$w_{4,1}$	$w_{5,4}$	1	0	1	0	0	0	0	0	0	0	0	0	0
$w_{4,2}$	$w_{5,1}$	2	2	1	2	3	0	0	2	2	1	0	0	0
$w_{4,2}$	$w_{5,2}$	5	3	4	4	2	2	3	3	2	0	1	2	1
$w_{4,2}$	$w_{5,3}$	4	5	1	3	5	1	3	2	2	1	1	1	1
$w_{4,2}$	$w_{5,4}$	1	0	3	1	0	2	1	1	0	0	0	1	0
$w_{4,3}$	$w_{5,1}$	2	1	2	1	1	0	0	1	0	0	0	0	0
$w_{4,3}$	$w_{5,2}$	1	1	3	3	0	2	2	1	1	0	0	2	1
$w_{4,3}$	$w_{5,3}$	3	1	2	1	1	2	2	2	1	0	1	1	0
$w_{4,3}$	$w_{5,4}$	0	0	1	1	0	0	0	0	0	0	0	2	1

u	v	$w \in W^{10}(8)$																
		$w_{10,1}$	$w_{10,2}$	$w_{10,3}$	$w_{10,4}$	$w_{10,5}$	$w_{10,6}$	$w_{10,7}$	$w_{10,8}$	$w_{10,9}$	$w_{10,10}$	$w_{10,11}$	$w_{10,12}$	$w_{10,13}$	$w_{10,14}$	$w_{10,15}$	$w_{10,16}$	$w_{10,17}$
$w_{1,1}$	$w_{9,1}$	1	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
$w_{1,1}$	$w_{9,2}$	0	1	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0
$w_{1,1}$	$w_{9,3}$	1	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0
$w_{1,1}$	$w_{9,4}$	0	1	0	1	1	0	0	0	0	0	1	0	0	0	0	0	0
$w_{1,1}$	$w_{9,5}$	0	0	0	0	0	1	1	0	0	0	0	0	1	0	0	0	0
$w_{1,1}$	$w_{9,6}$	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
$w_{1,1}$	$w_{9,7}$	0	1	0	0	0	0	0	1	1	0	0	0	0	1	0	0	0
$w_{1,1}$	$w_{9,8}$	0	0	0	0	0	1	0	0	0	1	1	0	0	0	0	0	0
$w_{1,1}$	$w_{9,9}$	0	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0	0
$w_{1,1}$	$w_{9,10}$	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0
$w_{1,1}$	$w_{9,11}$	0	0	1	0	0	0	0	0	0	0	0	1	0	1	0	0	0
$w_{1,1}$	$w_{9,12}$	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	1
$w_{1,1}$	$w_{9,13}$	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	1
$w_{2,1}$	$w_{8,1}$	1	2	1	0	0	2	2	0	0	0	0	0	1	0	0	0	0
$w_{2,1}$	$w_{8,2}$	1	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0
$w_{2,1}$	$w_{8,3}$	2	2	0	2	1	2	0	0	0	2	2	0	0	0	0	0	0
$w_{2,1}$	$w_{8,4}$	0	2	1	1	1	0	2	0	0	0	2	0	0	0	1	0	0
$w_{2,1}$	$w_{8,5}$	0	0	0	0	0	1	1	0	0	0	0	0	2	0	0	1	0
$w_{2,1}$	$w_{8,6}$	2	2	0	0	0	1	0	2	1	0	0	0	0	1	0	0	0
$w_{2,1}$	$w_{8,7}$	0	2	2	0	0	0	1	1	1	0	0	1	0	2	0	0	0
$w_{2,1}$	$w_{8,8}$	0	0	0	0	0	2	2	0	0	1	2	0	1	0	1	0	0
$w_{2,1}$	$w_{8,9}$	2	0	0	2	0	0	0	2	0	1	0	0	0	0	0	0	1
$w_{2,1}$	$w_{8,10}$	0	2	0	2	2	0	0	2	2	0	1	0	0	1	0	0	2
$w_{3,1}$	$w_{7,1}$	0	1	0	0	0	1	2	0	0	0	0	0	1	0	0	1	0
$w_{3,1}$	$w_{7,2}$	1	1	0	1	0	1	0	0	0	2	1	0	0	0	0	0	0
$w_{3,1}$	$w_{7,3}$	1	2	1	1	1	3	3	0	0	1	3	0	1	0	1	0	0
$w_{3,1}$	$w_{7,4}$	1	3	2	0	0	1	1	1	1	0	0	0	1	1	0	0	0
$w_{3,1}$	$w_{7,5}$	0	0	0	0	0	1	1	0	0	0	1	0	2	0	1	0	0
$w_{3,1}$	$w_{7,6}$	2	0	0	1	0	0	0	1	0	1	0	0	0	0	0	0	0
$w_{3,1}$	$w_{7,7}$	3	3	0	3	1	2	0	2	1	1	1	0	0	1	0	0	1
$w_{3,1}$	$w_{7,8}$	0	3	1	1	2	0	2	1	1	0	1	1	0	1	0	0	1
$w_{3,2}$	$w_{7,1}$	1	1	1	0	0	2	1	0	0	0	0	0	2	0	0	0	0
$w_{3,2}$	$w_{7,2}$	2	1	0	2	1	1	0	0	0	1	1	0	0	0	0	0	0
$w_{3,2}$	$w_{7,3}$	2	4	1	2	1	3	3	0	0	2	3	0	1	0	1	0	0
$w_{3,2}$	$w_{7,4}$	2	3	1	0	0	2	2	2	1	0	0	1	0	2	0	0	0
$w_{3,2}$	$w_{7,5}$	0	0	0	0	0	2	2	0	0	1	1	0	1	0	0	1	0
$w_{3,2}$	$w_{7,6}$	1	0	0	2	0	0	0	1	0	1	0	0	0	0	0	0	1
$w_{3,2}$	$w_{7,7}$	3	3	0	3	2	1	0	4	2	2	2	0	0	1	0	0	2
$w_{3,2}$	$w_{7,8}$	0	3	2	2	1	0	1	2	2	0	2	0	0	2	1	0	1
$w_{4,1}$	$w_{6,1}$	0	0	0	0	0	1	1	0	0	1	1	0	0	0	0	0	0
$w_{4,1}$	$w_{6,2}$	0	0	0	0	0	1	1	0	0	0	1	0	1	0	1	0	0
$w_{4,1}$	$w_{6,3}$	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0
$w_{4,1}$	$w_{6,4}$	1	1	0	1	0	1	0	0	0	1	0	0	0	0	0	0	0
$w_{4,1}$	$w_{6,5}$	1	2	1	1	1	1	1	0	0	0	1	0	1	0	0	0	0
$w_{4,1}$	$w_{6,6}$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$w_{4,2}$	$w_{6,1}$	2	3	1	2	1	3	2	0	0	2	3	0	1	0	1	0	0
$w_{4,2}$	$w_{6,2}$	1	3	1	1	1	4	5	0	0	1	3	0	3	0	1	1	0
$w_{4,2}$	$w_{6,3}$	1	3	2	0	0	2	2	1	1	0	0	0	2	1	0	0	0
$w_{4,2}$	$w_{6,4}$	5	3	0	4	1	2	0	3	1	3	2	0	0	1	0	0	1
$w_{4,2}$	$w_{6,5}$	4	9	3	4	3	5	5	4	3	2	4	1	1	3	1	0	2
$w_{4,2}$	$w_{6,6}$	1	0	0	1	0	0	0	1	0	1	0	0	0	0	0	0	0
$w_{4,3}$	$w_{6,1}$	2	2	0	2	1	1	1	0	0	1	1	0	0	0	0	0	0
$w_{4,3}$	$w_{6,2}$	2	2	1	1	0	3	1	0	0	2	1	0	1	0	0	0	0
$w_{4,3}$	$w_{6,3}$	2	1	0	0	0	2	1	1	0	0	0	1	0	1	0	0	0
$w_{4,3}$	$w_{6,4}$	1	1	0	3	2	0	0	2	1	1	1	0	0	0	0	0	2
$w_{4,3}$	$w_{6,5}$	3	4	1	3	1	1	1	4	2	2	3	0	0	2	1	0	1
$w_{4,3}$	$w_{6,6}$	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1
$w_{5,1}$	$w_{5,1}$	0	0	0	0	0	2	2	0	0	1	2	0	1	0	1	0	0
$w_{5,1}$	$w_{5,2}$	2	3	1	2	1	3	2	0	0	2	2	0	1	0	0	0	0
$w_{5,1}$	$w_{5,3}$	1	3	1	1	1	2	3	0	0	0	2	0	2	0	1	1	0
$w_{5,1}$	$w_{5,4}$	2	1	0	1	0	1	0	0	0	1	0	0	0	0	0	0	0
$w_{5,2}$	$w_{5,2}$	5	6	1	5	3	2	2	4	2	3	4	0	0	2	1	0	2
$w_{5,2}$	$w_{5,3}$	4	6	2	3	1	5	3	3	2	2	3	1	1	2	1	0	1
$w_{5,2}$	$w_{5,4}$	1	1	0	3	1	0	0	2	1	1	1	0	0	0	0	0	1
$w_{5,3}$	$w_{5,3}$	1	6	3	1	2	4	6	2	2	1	2	0	3	2	0	0	1
$w_{5,3}$	$w_{5,4}$	3	1	0	1	0	1	0	2	0	2	1	0	0	1	0	0	0
$w_{5,4}$	$w_{5,4}$	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	2

The computations were carried out by using Mathematicae on a PC. PIV667. Ram 128. Win98. In general, the running time of the program depends on

(1) the order of the coset $\overline{W}$;

(2) the number of non-zero entries in the Cartan matrix of G.

More precisely, to obtain the results in Table A$_n$, the times consumed (in seconds) are

n	6	7	8
time	1	1	2

.

The running times for computing all $a_{u,v}^w$ with $l(w) \leq 10$ are respectively

n	6	7	8
time	38	115	159

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